(8 pages)

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Code No. : 6380 **Sub. Code : ZMAM 34**

M.Sc. (CBCS) DEGREE EXAMINATION,
NOVEMBER 2022.

Third Semester

Mathematics — Core

TOPOLOGY — I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — ($10 \times 1 = 10$ marks)

Answer ALL questions.

Choose the correct answer :

1. Let $X = \{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a,ab\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b,cb\}\}$. The largest topology contained in τ_1 and τ_2 is

   (a) $\{\phi, X\}$

   (b) $\{\phi, X, \{ab\}\}$

   (c) $\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$

   (d) the discrete topology on X

2. Let $Y = (0,1]$ be a subspace of R. Let $A = \left(0, \frac{1}{2}\right)$ the closure of A in Y is

   (a) $(0, 1/2]$ (b) $[0, 1/2]$

   (c) $(0, 1/2)$ (d) $(0, 1]$

3. Let $f : \{a,b,c\} \to \{1,2,3\}$ be defined by $f(a) = 2, f(b) = 3$, $f(c) = 3$. Let $V = \{1,3\}$, then $f^{-1}(V)$ is

   (a) $\{a,b\}$ (b) $\{b,c\}$

   (c) $\{a,c\}$ (d) $\{a,b,c\}$

4. Consider the identity functions $f : R \to R_l$ and $g : R_l \to R$, then

   (a) both f and g are continuous

   (b) f is continuous but g is not continuous

   (c) f is not continuous but g is continuous

   (d) neither f nor g is continuous

5. In R^n, we have $d(x,y) \le \sqrt{n}\rho(x,y)$, this inequality shows that

(a) $B_\rho(x,\varepsilon/\sqrt{n}) \subset B_d(x,\varepsilon)$

(b) $B_\rho(x,\varepsilon) \subset B_d\left(x,\frac{\varepsilon}{\sqrt{n}}\right)$

(c) $B_\rho(x,\varepsilon) \subset B_d(x,\varepsilon)$

(d) $B_d(x,\varepsilon) \subset B_\rho\left(x,\frac{\varepsilon}{\sqrt{n}}\right)$

6. The metric that induces the product topology on R^ω is

(a) $D(x,y) = \sup\{\bar{d}(x_i,y_i)\}$

(b) $D(x,y) = \sup\{|x_n - y_n|\}$

(c) $D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$

(d) $D(x,y) = \sup\left\{\frac{i}{\bar{d}(x_i,y_i)}\right\}$

7. If X is connected then

(a) ϕ is the only subset which is both open and closed

(b) X is the only subset which is both open and closed

(c) ϕ and X are the only subsets which are both open and closed

(d) we can find a subset $A(\neq \phi, X)$ which is both open and closed

8. Which one of the following is compact in R

(a) [0, 1] (b) [0, 1)

(c) (0, 1) (d) (0, 1]

9. Which one of the following is NOT a subsequence of (x_n)

(a) $\{x_5, x_6, x_7, \ldots.\}$

(b) $\{x_3, x_6, x_9, \ldots.\}$

(c) $\{x_2, x_1, x_3, x_2, x_4, x_3 \ldots.\}$

(d) $\{x_{100}, x_{200}, x_{300}, \ldots\}$

10. Which one of the following is not locally compact

(a) R

(b) Q

(c) R^n

(d) Every simply ordered set having the $l.u.b.$ property

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Define a basis for a topology on X. If B is a basis for a topology τ on X, prove that τ equals the collection of all unions of elements AB.

Or

(b) Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ denote the closure of A in X. Prove that the closure of A in Y equals $\overline{A} \cap Y$.

12. (a) If $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology on Y, prove that $\mathcal{D} = \{B \times C / B \in \mathcal{B}, c \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Or

(b) State and prove the pasting lemma.

13. (a) Let X be a metric space with metric d. Define $\overline{d} : X \times x \to R$ by $\overline{d}(x, y) = \min\{d(x, y), 1\}$. Prove that $\overline{d}$ is a metric that induces the same topology as d.

Or

(b) State and prove the sequence lemma.

14. (a) Prove that the union of a collection of connected subspace of X that have a point in common is connected.

Or

(b) Prove that every closed subspace of a compact space is compact.

15. (a) Prove that compactness implies limit point compactness.

Or

(b) Let X be a Hausdorff space. Prove that X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b)

16. (a) Define the standard topology, the lower limit topology and the K - topology on R. Find the relation between these topologies.

Or

(b) Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Prove that $\overline{A} = A \cup A'$ and hence show that A is closed if and only if it contains all its limit points.

17. (a) Let X and Y be topological space; let $f : X \to Y$. Prove that the following are equivalent

(i) f is continuous

(ii) for every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$

(iii) for every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

Or

(b) Let $f : A \to \pi_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let πX_α have the product topology. Prove that the function f is continuous if and only if each function f_α is continuous.

18. (a) Prove that the topologies on R^n induced by the Euclidean metric d and the square metric ρ are the same as the product topology on R^n.

Or

(b) State and prove the uniform limit theorem.

19. (a) Prove that a finite Cartesian product of connected spaces is connected.

Or

(b) State and prove the tube lemma.

20. (a) Let X be a metrizable space. If X is sequentially compact, prove that X is compact.

Or

(b) Let X be a space. If X is locally compact Hausdorff prove that there exists a space Y satisfying

(i) X is a subspace of Y

(ii) $Y - X$ consists of a single point

(iii) Y is a compact Hausdorff space.